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On The Quaternionic Bertrand Curves In Semi-Euclidean 4-Space E_2^4

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ABSTRACT: *In this paper, we define characterizations of semi-real quaternionic Bertrand curves in the four-dimensional semi-Euclidean space E_2^4 . In order to do this, we study the Serret-Frenet formulae of the curve in E_2^4 and then applying quaternionic Bertrand curves. We obtain with vanishing curvatures form the families of quaternionic Bertrand curves in E_2^4.*
Keywords -*Semi-Euclidean spaces, Quaternion algebra, Quaternionicframe,Quaternionic Bertrand curves*

I. INTRODUCTION

The quaternions first described by Sir William R. Hamilton in 1843 as a number system that extends the complex numbers. Whereas a standard complex number has a scalar component and an imaginary component, with quaternions the imaginary part is an imaginary vector based on three imaginary orthogonal axes. The quaternions are both relatively simple and very effective for rotations. So, the quaternion algebra has played a significant role recently in several areas of the physical science; namely, in differential geometry, in analysis and synthesis of mechanism and machines, simulation of particle motion in molecular physics and quaternionic formulation of equation of motion in theory of relativity,[1, 2].

In 1888, C. Bioche give a new theorem in [3] to obtaining Bertrand curves by using the given two curves C_1 and C_2 in Euclidean 3-space. Later, in 1960, J. F. Burke give a theorem related with Bioche's thorem on Bertrand curves,[4].

In 1987, The Serret-Frenet formulae for a quaternionic curves in $\mathbb{R}^3$ are introduced by K. Bharathi and M. Nagaraj. Moreover, they obtained the Serret-Frenet formulae for the quaternionic curves in $\mathbb{R}^4$by the formulae in $\mathbb{R}^3$, [5]. Then, lots of studies have been published by using this studies. One of them is A. C. Çöken and A. Tuna's study,[6,7] which they gave Serret-Frenet formulas, inclined curves, harmonic curvatures and some characterizations for a quaternionic curve in the semi- Euclidean spaces E_2^4 .

In this work, we define characterizations of semi-real quaternionic Bertrand curves in the four-dimensional semi-Euclidean space E_2^4 . In order to do this, we study the Serret-Frenet formulae of the curve in $E_2^4$and then applying quaternionic Bertrand curves. We obtain with vanishing curvatures form the families of quaternionic Bertrand curves in E_2^4.

II. PRELIMINARIES

Let Q_v be the four-dimensional vector space over a field v whose characteristic greater than 2. Let e_i ($1\leq i \leq 4$) be a basis for the vector space. Let the rule of multiplication on Q_v be defined on e_i and extended to the whole of the vector space distributivity as follows [6]:

A real quaternion is defined by $q = ae_1 + be_2 + ce_3 + d$ (or $S_q = d$ and $V_q = ae_1 + be_2 + ce_3$). Then a quaternion q can now write as $q = S_q + V_q$, where S_q and V_q are the scalar part and vectorial part of q, respectively. we define the set of all real quaternions by

$$Q_v = \{ q \mid q = ae_1 + be_2 + ce_3 + d \,; a, b, c, d \in R and\ e_1, e_2, e_3 \in R^3 \}.$$

Using these basic products, we can now expand the product of two quaternions to give

$p \times q = S_p S_q + \langle V_p , V_q \rangle + S_p V_q + S_q V_p + V_p \wedge V_q$ for every $p, q \in Q_v$. Where we have used the dot and cross products in Euclidean space E^4. We see that the quaternionic product contains all the products of Euclidean space E^4. There is a unique involutory antiautomorphism of the quaternion algebra, denoted by the symbol $\gamma$and defined as follows:

$$\gamma q = -ae_1 - be_2 - ce_3 \text{ for every } q = ae_1 + be_2 + ce_3 + d \in Q_v.$$

which is called the "Hamiltonian conjugation". This h-inner product of two quaternions is define by

$$\mathrm{h}(p, q) = \frac{1}{2}[(p \times \gamma q) + (q \times \gamma p)]\ for every p, q \in Q_v$$

where h is the symmetric, non-degenerate, real valued and bilinear form. The norm of real quaternion q is denoted by

$$\|q\|^2 = |h_v(q,q)| = |(q \times \gamma q)| = |\, a^2 + b^2 + c^2 + d^2|$$

for $p, q \in Q_v$ where if $h_v(p,q) = 0$ then p and q are called h-orthogonal.The concept of a spatial quaternion will be made use throughout our work.q is called a spatial quaternion whenever$q + \gamma q = 0$ [6, 7].

III. SERRET-FRENET FORMULAE FOR QUATERNIONIC CURVES IN $\mathbb{E}_2^4$

Definition1: The 4-dimensional semi-Euclidean spaces in $\mathbb{E}_2^4$ are identified with the spaces of unit quaternions. Let $\alpha: I \subset \mathbb{R} \longrightarrow \mathbb{E}_2^4$ be a unit speed quaternionic curve defined over the interval $I = [0, 1]$ and the arc-length parameter s be chosen such that the tangent $T = \alpha^{'}(s)$ has unit magnitude. The Serret-Frenet apparatus of α can be written as follows:
The four-dimensional Euclidean spaces in $\mathbb{E}_2^4$ are identified with the spaces of unit quaternions. Let

$$\alpha : I \subset \mathbb{R} \rightarrow Q_v$$

$$s \rightarrow \alpha(s) = \sum_{i=1}^{4} \alpha_i(s) e_i, 1 \leq i \leq 4, e_4 = 1$$

be a smooth curve in $\mathbb{E}_2^4$. Let the parameter s be chosen such that the tangent $T(s) = \alpha'(s)$ has unit magnitude. Let $\{T; N; B; B_1\}$ be the Frenet apparatus of the differentiable Euclidean space curve in the Euclidean spaces E^4. Then Frenet formulas are given by

$$\left\{\begin{array}{l} T'(s) \ = \ \varepsilon_N \kappa(s) N(s) \\ N'(s) \ = \ \varepsilon_n \tau(s) B(s) - \varepsilon_N \varepsilon_t \kappa(s) T(s) \\ B^{'}(s) \ = \ -\varepsilon_t \tau(s) N(s) + \varepsilon_n [\, \sigma - \kappa \varepsilon_N \varepsilon_T \varepsilon_t](s) B_1(s) \\ B_1^{'}(s) \ = \ -\varepsilon_B [\sigma - \kappa \varepsilon_N \varepsilon_T \varepsilon_t](s) B(s). \end{array}\right\} (1)$$

Where $\kappa(s) = \varepsilon_N \|T'(s)\|$ and $N^{'}(s) = |\varepsilon_N|$

IV. QUATERNIONIC BERTRAND CURVES IN SEMI-EUCLIDEAN SPACE E_2^4

In this section, we give the definition and characterizations of the four-dimensional quaternionic Bertrand curves in $\mathbb{E}_2^4$and investigate the properties of it.

Definition 2: Let $\mathbb{E}_2^4$ be be the four-dimensional semi -Euclidean space with inner product $h_v(p,q)$ for every $p, q \in Q_v$. If there exists a corresponding relationship between the quaternionic curves α and α^*. such that , at the corresponding points of the quaternionic curves,the principal normal vector of α coincides with normal vector of α^*, then α is called a Bertrand curve, and α^*a Bertrand partner curve of α. The pair $\{\alpha, \alpha^*\}$ is said to be a Bertrand pair.

Definition 3: Let $\alpha(s)$ and $\alpha^*(s^*)$be quaternionic curves in $\mathbb{E}_2^4$ with arc-length parameter s and s^*, respectively. $\{T(s), N(s), B(s), B_1(s)\}$ and $\{T^*(s^*), N^*(s^*), B^*(s^*), {B_1}^*(s^*)\}$ are Frenet frames of α and α^*, respectively. If the pair $\{\alpha, \alpha^*\}$ are a Bertrand pair, $N(s)$ and $N^*(s^*)$ are linearly dependent. We can write

$$\alpha^*(s) = \alpha(s) + \lambda(s) N(s). \qquad (2)$$

Theorem 1: Let α(s) be a quaternionic curves in semi -Euclidean space $\mathbb{E}_2^4$with arc-length parameter s. Then $\alpha^*(s)$ is the Bertrand partner curve of $\alpha(s)$, Then corresponding points are a fixed distance for all s∈I.

Proof: Suppose that α(s):I⊂R→$\mathbb{E}_2^4$is a quaternionic curve.
Taking the derivate of (2) with respect to s and applyig the Frenet formulas, we obtain

$$\frac{d\alpha^*(s^*)}{ds^*}\frac{ds^*}{ds} = \ [1 - \varepsilon_N \varepsilon_t \lambda(s)\kappa(s)]T(s) + \lambda^{'}(s)N(s) + \varepsilon_n \, \tau(s)B(s)$$

$$T^*(s^*) \ = \frac{ds}{ds^*} \, ([1 - \varepsilon_N \varepsilon_t \lambda(s)\kappa(s)]T(s) + \lambda^{'}(s)N(s) + \varepsilon_n \, \tau(s)B(s))$$

And

$$h(T^*(s^*), N^*(s^*)) = \left\{\begin{array}{c} \dfrac{ds}{ds^*} \, ([1 - \varepsilon_N \varepsilon_t \lambda(s)\kappa(s)] h(T(s), N^*(s^*)) \\ +\lambda^{'}(s) h(N(s), N^*(s^*)) + \varepsilon_n \, \tau(s) h(B(s), N^*(s^*)) \end{array}\right\}$$

Since $N(s)$ is coincident with $N^*(s^*)$ in direction " a linearly dependent set " ,we get$\lambda'(s) = 0$.This means that λ(s) is a none zero constant function on I. Then

$$d(\alpha^*(s), \alpha(s)) = \|\alpha^*(s) - \alpha(s)\| = \|\lambda(s)N(s)\| = \lambda.$$

Where $\lambda =$ constant. This completes the proof.

Theorem 2: Let $\alpha(s)$ be a quaternionic curves in $\mathbb{E}_2^4$with arc-length parameter s. and $\alpha^*(s^*)$ is a Bertrand partner curve of $\alpha(s)$**.** Then the angle between the tangent vector of quaternionic curve $\alpha(s)$ and $\alpha^*(s)$ is constant.

Proof: Let $\alpha(s)$ and $\alpha^*(s^*)$ be quaternionic curves in $\mathbb{E}_2^4$ with arc-length s and s^*, respectively. Let's consider that

$$\mathrm{h}(T(s), T^*(s^*)\ ) = cos\theta. \ (3)$$

Taking the derivate of (3) with respect to s , we obtain

$$\frac{\mathrm{d}}{\mathrm{ds}}\mathrm{h}(\mathrm{T(s)}, T^*(s^*)) \ = \mathrm{h}\left(\frac{\mathrm{dT(s)}}{\mathrm{ds}}, T^*(s^*)\right) + \mathrm{h}(\ \mathrm{T(s)}, \frac{\mathrm{d}T^*(s^*)}{\mathrm{ds}}\frac{\mathrm{d}s^*}{\mathrm{ds}})$$

$$= \mathrm{h}\big(\varepsilon_N\varepsilon_t\kappa(s)N(s), T^*(s^*)\big) + \mathrm{h}\left(\ \mathrm{T(s)}, \varepsilon_N\varepsilon_t\kappa^*(s^*)N^*(s^*)\frac{\mathrm{d}s^*}{\mathrm{ds}}\right) = 0.$$

This means that $\mathrm{h}(\mathrm{T(s)}, T^*(s^*))$ =constant. This completes the proof**.**

Theorem 3: Let $\alpha(s)$ and $\alpha^*(s)$ be quaternionic curves in $\mathbb{E}_2^4$ with arc-length parameter s and s^*, respectively. Then α is a quaternionic Bertrand curve if and only if

$$\lambda\varepsilon_N\varepsilon_t\kappa(s) + \mu\varepsilon_n\,\tau(s) = 1,$$

where λ and μ real constants and $\kappa(s)$ is the principal curvature, $\tau(s)$ is the torsion of the curve α.

Proof: Let $\alpha^*(s^*)$ be a quaternionic Bertrand partner curve of $\alpha(s)$**.** Then we can write

$$\alpha^*(s) = \alpha(s) + \lambda(s)N(s).$$

Taking the derivative of the last equality considering$\psi: I \to I^*, \psi(s) = s^*$ is a C^∞ function we have

$$T^*(\psi(s)) = \frac{ds}{ds^*}[[1 - \varepsilon_N\varepsilon_t\lambda(s)\kappa(s)]T(s) + \varepsilon_n\lambda(s)\tau(s)B(s).$$

If we consider the following equation

$$T^*(\psi(s)) = cos\theta T(s) + sin\theta B(s)$$

we get

$$cos\theta \ = \ (1 - \varepsilon_N\varepsilon_t\lambda(s)\kappa(s))\frac{ds}{ds^*}$$

$$sin\theta \ = \ \varepsilon_n\,\tau(s)\lambda(s)\frac{ds}{ds^*}.$$

Then by taking $\lambda\frac{COS\varphi}{sin\varphi} = \mu$, we have

$$\lambda\varepsilon_N\varepsilon_t\kappa(s) + \mu\varepsilon_n\,\tau(s) = 1.$$

On the other hand , Suppose that $\alpha(s)$ be quaternionic curves in $\mathbb{E}_2^4$with curvature function $\kappa, \tau \neq 0$ satisfiying the relation { $\lambda\varepsilon_N\varepsilon_t\kappa(s) + \mu\varepsilon_n\tau(s) = 1$} for constant λ, μ . Then we can write

$$\alpha^*(s) = \alpha(s) + \lambda(s)N(s).$$

By taking the derivative of the last equality with respect to s and applying the Frenet formulas , we have

$$\frac{d\alpha^*(s^*)}{ds^*}\frac{ds^*}{ds} = \ [1 - \varepsilon_N\varepsilon_t\lambda(s)\kappa(s)]T(s) + \varepsilon_n\lambda(s)\,\tau(s)B(s)$$

$$= \frac{ds}{ds^*}[[1 - \varepsilon_N\varepsilon_t\lambda(s)\kappa(s)]T(s) + \varepsilon_n\lambda(s)\tau(s)B(s)$$

From the hypothesis , we have $\mu\varepsilon_n\,\tau(s) = 1 - \lambda\varepsilon_N\varepsilon_t\kappa(s)$thus we get

$$\frac{d\alpha^*(s^*)}{ds^*} = \ T^*(s^*) = \frac{ds}{ds^*}[\varepsilon_n\,\tau(s)][\mu T(s) + \lambda B(s)]$$

Where,

$$\frac{ds}{ds^*} \ = \frac{cos\theta}{\mu\varepsilon_n\,\tau(s)}\ ;\ \mu^2(\frac{ds}{ds^*})^2 = \frac{cos^2\theta}{(\varepsilon_n\,\tau(s))^2},$$

$$\frac{ds}{ds^*} = \frac{sin\theta}{\lambda\varepsilon_n\tau(s)}\ ;\ \lambda^2(\frac{ds}{ds^*})^2 = \frac{sin^2\theta}{(\varepsilon_n\,\tau(s))^2}.$$

$$(\mu^2 + \ \gamma^2)\left(\frac{ds}{ds^*}\right)^2 = \frac{1}{(\varepsilon_n\,\tau(s))^2}\text{and}\ \frac{ds}{ds^*} = \pm\frac{\varepsilon_n\,\tau(s)}{|\varepsilon_n\,\tau(s)|\sqrt{\lambda^2+\mu^2}}$$

we have

$$T(s^*) = \pm\frac{\varepsilon_n\,\tau(s)}{|\varepsilon_n\,\tau(s)|\sqrt{\lambda^2+\mu^2}}(\ \mu T(s) + \lambda B(s))$$

Differentiating of the last equality with respect to s and applying the Frenet formulas, we have

$$\varepsilon_N \kappa^*(s^*) N^*(s^*) = \pm \frac{\varepsilon_n \, \tau(s)}{|\varepsilon_n \, \tau(s)| \sqrt{\lambda^2 + \mu^2}} \left(\mu \varepsilon_N \kappa(s) - \lambda \varepsilon_t \, \tau(s) \right) N(s).$$

Thus, we obtain $N^*(s^*)$ and $N(s)$ are linearly dependent.

Theorem 4: Let α and α^* be a quaternionic curves in $\mathbb{E}_2^4$with arc-length parameter s and s^*, respectively. If $\alpha^*(s^*)$ is a quaternionic Bertrand partner curve of $\alpha(s)$. Then $(\tau(s).\tau^*(s^*))$ is constant and

$$\varepsilon_n \mu(\tau + \tau^*) + \varepsilon_t \varepsilon_N \lambda(\kappa + \kappa^*) = 0.$$

Proof: Let $\alpha^*(s^*)$ be a quaternionic Bertrand partner curve of $\alpha(s)$.Then we can write

$$\alpha^*(s) = \alpha(s) + \lambda(s) N(s).$$

If we interchange the position of curves α(s) and $\alpha^*(s^*)$, we can write

$$\alpha(s) = \alpha^*(s^*) - \lambda(s^*) N^*(s^*)$$

Differentiating of the last equality with respect to s and applying the Frenet formulas , we have

$$\mathrm{T}(s) \;=\; [1 - \varepsilon_N \varepsilon_t \lambda(s) \kappa^*(s^*)] T^*(s^*) + \varepsilon_n \lambda(s) \tau^*(s^*) B^*(s^*)$$

$$T(s) \;=\; (\,cos\theta) T^*(s^*) + (sin\theta) B^*(s^*)$$

we have

$$cos\theta \;=\; [1 - \varepsilon_N \varepsilon_t \lambda(s) \kappa^*(s^*)] \frac{ds^*}{ds}$$

$$sin\theta \;=\; -\lambda \varepsilon_n \tau^*(s^*) \frac{ds^*}{ds}.$$

where multiplying both sides of the last equality with

$cos\theta \;=\; [1 - \varepsilon_N \varepsilon_t \lambda(s) \kappa^*(s^*)] \frac{ds^*}{ds}$ and $sin\theta \;=\; -\lambda \varepsilon_n \tau^*(s^*) \frac{ds^*}{ds}$, we obtain

$$\tau\tau^* \;= -\frac{sin^2\theta}{(\varepsilon_n \lambda)^2} \;= constant.$$

$$\varepsilon_n \mu(\tau + \tau^*) + \varepsilon_N \varepsilon_t \lambda\, (\kappa + \kappa^*) \;= 0$$

This completes the proof.

Example: We consider a quaternionic curve with arc-length parameter s, $\alpha(s): I \subset R \to \mathbb{E}_2^4$,

$\alpha(s) = (coshs, \sqrt{2}s, sinhs, \sqrt{2})$for all $s \in I$. The curve α is regular curve and its curvature function $\kappa = -1, \tau = \sqrt{2}\ and\ \sigma - \kappa \varepsilon_N \varepsilon_T \varepsilon_t = 0$. For

$$\lambda = \frac{1}{\varepsilon_N \varepsilon_t}, \mu = \frac{\sqrt{2}}{\varepsilon_n}$$

curvature of $\alpha(s)$ curve satisfy the relation $\lambda \varepsilon_N \varepsilon_t \kappa(s) + \mu \varepsilon_n \, \tau(s) = 1$. So, $\alpha(s)$ is a quaternionic Bertrand curve and we have its quaternionic Bertrand partner $\alpha^*(s)$ as follows

$$\alpha^*(s) = (\sqrt{2}coshs, 2s, \sqrt{2}sinhs, \sqrt{2}).$$

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REFERENCES

[1]. Hamilton, W.R., Element of Quaternions, I, II and III, chelsea, New YORK, 1899.

[2]. B. O Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press Inc., London, 1983.

[3]. Ch.Bioche, Sur les courbes de M. Bertrand, Bull. Soc. Math. France 17 (1889), 109-112.

[4]. John F. Burke, Bertrand Curves Associated with a Pair of Curves Mathematics Magazine, Vol. 34, No. 1 (Sep.-Oct.,1960), pp. 60-62

[5]. K. Bharathi, M. Nagaraj, Quaternion valued function of a real Serret-Frenet formulae, Indian J. Pure Appl. Math. 16 (1985) 741-756.

[6]. A. C. Çöken, A. Tuna, On the quaternionic inclined curves in the semi-Euclidean space $\mathbb{E}_2^4$: Applied Mathematics and Computation 155 (2004), 373-389.

[7]. A. Tuna, Serret Frenet Formulae for Quaternionic Curves in Semi Euclidean Space, Master Thesis, Süleyman Demirel UniversityGraduate School of Natural and Applied Science,Department of Mathematics, Isparta, Turkey, 2002.

[8]. J.P. Ward, Quaternions and Cayley Numbers, Kluwer Academic Publishers, Boston/London,1997.